

## IQI 04, Seminar 2

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- Qubits

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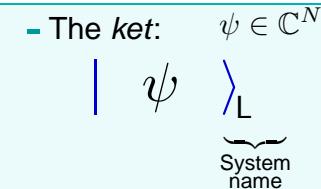


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## Qubits' State Space I

- Two qubits, labeled A and B.
  - State space of qubit A:  
 $\{\alpha|0\rangle_A + \beta|1\rangle_A \mid |\alpha|^2 + |\beta|^2 = 1\}$ .
  - State space of qubit B:  
 $\{\alpha|0\rangle_B + \beta|1\rangle_B \mid |\alpha|^2 + |\beta|^2 = 1\}$ .
  - Some joint states, the *logical states*:  
 $|0\rangle_A|0\rangle_B, |0\rangle_A|1\rangle_B, |1\rangle_A|0\rangle_B, |1\rangle_A|1\rangle_B$ .  
Notation:  $|\psi\rangle_A|\phi\rangle_B \doteq |\phi\rangle_B|\psi\rangle_A \doteq |\psi\phi\rangle_{AB} \doteq |\phi\psi\rangle_{BA}$ .
  - Qubits A and B together form the system AB with state space all superpositions of the logical states.  
$$\{\alpha|00\rangle_{AB} + \beta|01\rangle_{AB} + \gamma|10\rangle_{AB} + \delta|11\rangle_{AB} \mid |\alpha|^2 + |\beta|^2 + |\gamma|^2 + |\delta|^2 = 1\}$$

- The ket:  $\psi \in \mathbb{C}^N$



$|\psi\rangle_L$   
System name

## Qubits' State Space II

- Vector representation of AB's state space.

$$\alpha|00\rangle_{AB} + \beta|01\rangle_{AB} + \gamma|10\rangle_{AB} + \delta|11\rangle_{AB} \leftrightarrow \begin{pmatrix} \alpha \\ \beta \\ \gamma \\ \delta \end{pmatrix} \quad \text{lexicographic ordering}$$

- Examples: The logical states,

- $\frac{1}{2}|00\rangle_{AB} + \frac{1}{2}|01\rangle_{AB} + \frac{1}{2}|10\rangle_{AB} + \frac{1}{2}|11\rangle_{AB}$   
 $= \left(\frac{1}{\sqrt{2}}|0\rangle_A + \frac{1}{\sqrt{2}}|1\rangle_A\right) \left(\frac{1}{\sqrt{2}}|0\rangle_B + \frac{1}{\sqrt{2}}|1\rangle_B\right)$

- Product states:  $(a|0\rangle_A + b|1\rangle_A)(c|0\rangle_B + d|1\rangle_B)$

- A Bell state:  $\frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$  Is this a product state?

- Global phase:  $|\psi\rangle_{AB} \simeq e^{i\phi}|\psi\rangle_{AB}$

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## Qubits' State Space III

- State space of three qubits A, B, C:  
Superpositions of the 8 logical states  $|\alpha\beta\gamma\rangle_{ABC}$ .
- State space of  $n$  qubits 1, 2, ..., n:  
Superpositions of the  $2^n$  logical states  $|\alpha_1\alpha_2\dots\alpha_n\rangle_{12\dots n}$ .
- Can one take superpositions of other states to form the state space?  
If the states are *distinguishable*. ... Explanation deferred.

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## Ket Algebra

- Ket algebra: Addition and multiplication rules.

- Kets for completely separate systems commute.

$$|\psi\rangle_S |\phi\rangle_L = |\phi\rangle_L |\psi\rangle_S.$$

... we'll avoid multiplying kets of the same system.

- Products of atomic kets may be merged to abbreviate.

$$|\psi\rangle_S |\phi\rangle_L = |\psi\phi\rangle_{SL}.$$

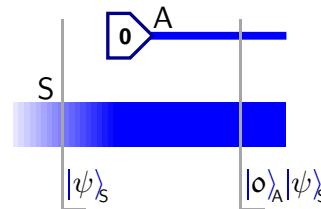
- Multiplication distributes over addition.

$$\begin{aligned} \frac{1}{\sqrt{2}} |1\rangle_A (|0\rangle_B + i|1\rangle_B) |0\rangle_D &= \frac{1}{\sqrt{2}} |1\rangle_A |0\rangle_B |0\rangle_D + \frac{i}{\sqrt{2}} |1\rangle_A |1\rangle_B |0\rangle_D \\ &= \frac{1}{\sqrt{2}} |100\rangle_{ABD} + \frac{i}{\sqrt{2}} |110\rangle_{ABD} \end{aligned}$$

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## State Preparation

- Prepare a new qubit A in  $|0\rangle_A$ ,  $\text{prep}(o)^{(A)}$ .



- Label A must not have been used previously.

- Similarly, can prepare  $|1\rangle_A$  using  $\text{prep}(o)^{(A)}$ .

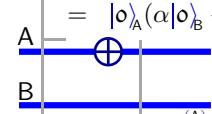
- Notation:  $\text{op}^{(S)}$  means op acts on system S.

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## One-Qubit Gates

- Apply  $\text{not}^{(A)}$  to the two-qubit system AB.

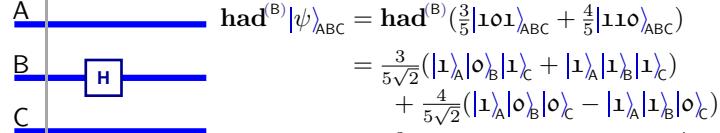
$$|\psi\rangle_{AB} = \alpha|00\rangle_{AB} + \beta|01\rangle_{AB} + \gamma|10\rangle_{AB} + \delta|11\rangle_{AB}$$



$$\begin{aligned} \text{not}^{(A)}|\psi\rangle_{AB} &= (\text{not}^{(A)}|0\rangle_A)(\alpha|0\rangle_B + \beta|1\rangle_B) + (\text{not}^{(A)}|1\rangle_A)(\gamma|0\rangle_B + \delta|1\rangle_B) \\ &= |1\rangle_A(\alpha|0\rangle_B + \beta|1\rangle_B) + |0\rangle_A(\gamma|0\rangle_B + \delta|1\rangle_B) \\ &= \alpha|10\rangle_{AB} + \beta|11\rangle_{AB} + \gamma|00\rangle_{AB} + \delta|01\rangle_{AB} \end{aligned}$$

- Apply  $\text{had}^{(B)}$  to the three-qubit system ABC in state

$$|\psi\rangle_{ABC} = \frac{3}{5}|101\rangle_{ABC} + \frac{4}{5}|110\rangle_{ABC}.$$

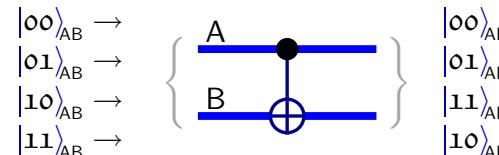


$$\begin{aligned} \text{had}^{(B)}|\psi\rangle_{ABC} &= \text{had}^{(B)}(\frac{3}{5}|101\rangle_{ABC} + \frac{4}{5}|110\rangle_{ABC}) \\ &= \frac{3}{5\sqrt{2}}(|1_A|0\rangle_B|1\rangle_C + |1_A|1\rangle_B|1\rangle_C) \\ &\quad + \frac{4}{5\sqrt{2}}(|1_A|0\rangle_B|0\rangle_C - |1_A|1\rangle_B|0\rangle_C) \\ &= \frac{3}{5\sqrt{2}}(|101\rangle_{ABC} + |111\rangle_{ABC}) + \frac{4}{5\sqrt{2}}(|100\rangle_{ABC} - |110\rangle_{ABC}) \end{aligned}$$

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## Controlled-Not

- The controlled not acts on two qubits: if A then  $\text{not}^{(B)}$ .



$$\text{cnot}^{(AB)}|ab\rangle_{AB} = |a(b+a)\rangle_{AB} \dots " + " \text{ is modulo 2: } \begin{cases} 0+0 = 0 \\ 0+1 = 1 \\ 1+0 = 1 \\ 1+1 = 0 \end{cases}$$

- cnot acts on superpositions by linear extension of above.

- Example:

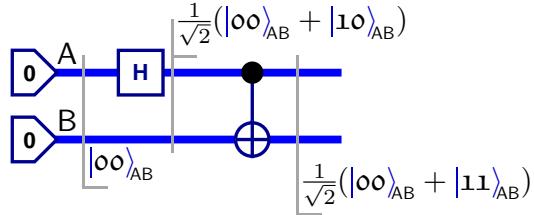
$$\begin{aligned} \text{cnot}^{(AS)}(\dots + \beta|010\rangle_{SAB} + \gamma|100\rangle_{SAB} + \dots) \\ = (\dots + \beta|110\rangle_{SAB} + \gamma|100\rangle_{SAB} + \dots) \end{aligned}$$

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## Using the Controlled Not

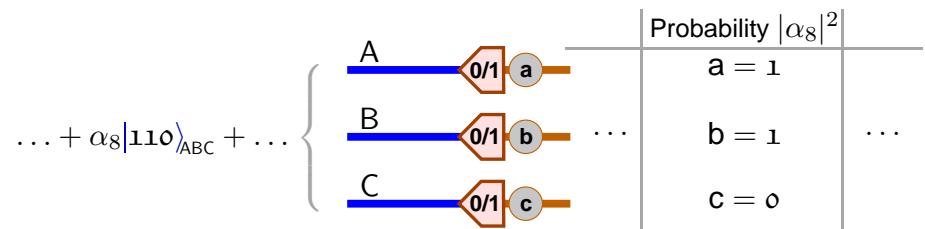
- Problem: Prepare the state  $\frac{1}{\sqrt{2}}(|00\rangle_{AB} + |11\rangle_{AB})$ .

- Solution:

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## Measuring Qubits

- Measuring all qubits at once.

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## Operators and Kets

- Ket algebra: Rules involving operators.

- Operators and kets for disjoint systems commute.

$$\text{op}^{(x)} |\psi\rangle_y = |\psi\rangle_y \text{op}^{(x)}$$

- Operator multiplication distributes over sums.

$$\text{op}^{(x)} (\alpha |\psi\rangle_{SXY} + \beta |\phi\rangle_{SXY}) = \alpha \text{op}^{(x)} |\psi\rangle_{SXY} + \beta \text{op}^{(x)} |\phi\rangle_{SXY}$$

- An operator on S applied to an S-ket expression results in a ket expression using the defined action of the operator.

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## A Black Box Problem

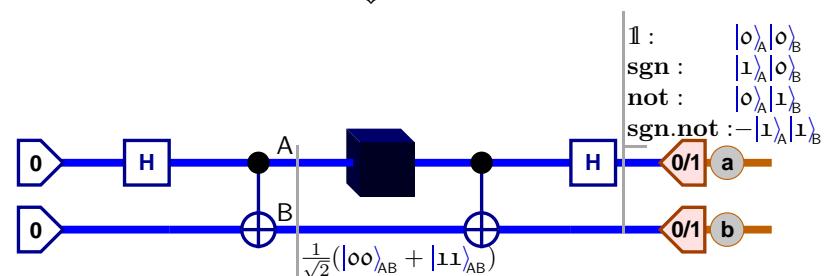
- Given: Unknown one-qubit device, a “black box”.

Promise: It either applies **not**, **sgn**, **sgn.not** or does nothing.

Problem: Determine which using the device once.

- Solution, using two qubits.

$$\underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}}$$

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